MODAL LOGIC 3.1 — QUANTIFIED MODAL LOGIC LOA 26/5/3

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1. Preliminaries

◆ Vocabulary:

— variables x_0, x_1, x_2, ...
— names a_0, a_1, a_2, ...
— predicates $\mathbb{P}_0$, $\mathbb{P}_1$, $\mathbb{P}_2$, ...
— connectives and quantifiers , ,

◆ Grammar:

— Straightforward

— But notice that the interplay between quantifiers and modalities yields *de dicto/de re* readings:

de dicto:	$\Box\ \ xA$	e.g., *Necessarily, everything is spatio-temporally located.*
de re:	$x\Box A$	e.g., *Everything is necessarily spatio-temporally located.*
de dicto:	$\Diamond\ \ xA$	e.g., *It is possible that everybody votes for Berlusconi.*
de re:	$x\Diamond A$	e.g., *Anybody could vote for Berlusconi.*
de dicto:	$\Box\ \ xA$	e.g., *There must necessarily be right-winged politicians.*
de re:	$x\Box A$	e.g., *Somebody is necessarily a right-winged politician.*
de dicto:	$\Diamond\ \ xA$	e.g., *Bush might have had a sister.*
de re:	$x\Diamond A$	e.g., *Somebody might have been Bush's sister.*

◆ Semantics:

— Two main approaches.

1. The <u>possibilist</u> (or fixed-domain) approach: each model comes with a single domain of quantification containing all possible objects (i.e., all objects that are possible according to the model).
2. The <u>actualist</u> (or world-relative) approach: each model comes with a domain of quantification for each world in the model—a domain containing only those objects that actually exist in the given world.

— We shall look at these options only from the perspective of Kripke-style semantics, but Montague-style semantics admit of a similar distinction.

2. Semantics # 1: The fixed-domain approach

◆ MODELS

A model is a four-tuple $\mathcal{M} = \;\; W, R, D, V\;\;$ such that:

— W is a non-empty set (of worlds);
— R is a binary (accessibility) relation on W;
— D is another non-empty set (of individuals);
— V is a function assigning
  - an element $V(a) \quad D$ to every name a.
  - a relation $V(\mathbb{P}) \quad W \times D^n$ to every *n-ary* predicate $\mathbb{P}$. (If $\mathbb{P}$ is 0-ary, $V(\mathbb{P})$ is just the *proposition P* expressed by $\mathbb{P}$, as before; otherwise $V(\mathbb{P})$ is the *intension* of $\mathbb{P}$.)

In such a model, an assignment μ is a function that maps every variable x to a value $\mu(x) \quad D$, and two assignments μ and μ' are said to be x-alternatives iff they agree on every variable except possibly for x [i.e., iff $\mu(y)=\mu'(y)$ for every variable y distinct from x].

◆ TRUTH AND VALIDITY

The truth conditions of a wff A at a world α under an assignment μ are defined as follows, where $\mu^*(t)=\mu(t)$ if 't' is a variable, and $\mu^*(t)=V(t)$ if 't' is a constant

1. $\models^{\mathcal{M}}_{\alpha} \mathbb{P}t_1 ... t_n\ [\mu]$ iff $\alpha, \mu^*(t_1), ..., \mu^*(t_n) \quad V(\mathbb{P})$
2. $\models^{\mathcal{M}}_{\alpha} \neg A\ [\mu]$ iff not $\models^{\mathcal{M}}_{\alpha} A\ [\mu]$
3. $\models^{\mathcal{M}}_{\alpha} A \quad B\ [\mu]$ iff $\models^{\mathcal{M}}_{\alpha} A\ [\mu]$ and $\models^{\mathcal{M}}_{\alpha} B\ [\mu]$
4. $\models^{\mathcal{M}}_{\alpha} \Box A\ [\mu]$ iff $\models^{\mathcal{M}}_{\beta} A\ [\mu]$ for every $\beta \quad W$ such that $\alpha R \beta$.
5. $\models^{\mathcal{M}}_{\alpha} \quad xA\ [\mu]$ iff $\models^{\mathcal{M}}_{\alpha} A\ [\mu']$ for every x-alternative of μ' of μ

— A wff A is true at α in $\mathcal{M}$ iff $\models^{\mathcal{M}}_{\alpha} A\ [\mu]$ for every assignment μ.
— The other semantic notions (validity in a model, validity in a class C of models, and logical validity) are defined as before.

◆ REMARK 1

— The semantic condition on names corresponds to the idea that names are *rigid designators.*

◆ REMARK 2

— Concerning predicates, there is an expressiveness problem. Consider:

$\Diamond \quad xRx$ e.g., *There is a world in which everybody is rich*

— This formula refers to those existing at some world and says that they are all rich.
But do we mean to say that at some world everybody will be rich as <u>they</u> use the word 'rich', or as <u>we</u> use the word in the actual world?

— It could be argued that the above semantics is constrained to interpret wffs involving '*R*' at a world according to that world's meaning of '*R*', so it only gives us the first option. But this is not quite right.

— The same problem arises with *de re* modalities:

$x\Diamond Rx$ e.g., *Everybody could be rich.*

We certainly don't mean to say that everybody could be rich insofar as 'rich' could have a different meaning...

◆ LOGIC

— Modal logics adequate to this semantics can be axiomatized by supplementing the principles of whichever propositional modal logic one chooses with the classical quantifier rules together with the so-called Barcan Formula (Barcan 1946):

BF $x\Box A \rightarrow \Box\ \ xA$

— This is actually provable as a theorem in quantified *KB* as also in quantified *KT5*.)

◆ REMARK 3

— **BF** captures one half of the "fixed domain" idea: it tells us that the domain of every possible world is included in the domain of the given world. [Otherwise **BF** could be false, for even if everything falling under the actual range of the quantifiers were *A* in every world, there might be something *else* at some world that failed to be *A* at that world.]

— The other half of the "fixed domain" idea is captured by the converse of **BF**, which is valid (and provable in quantified *K* even without **BF**).

$\mathbf{BF_c}$ $\Box\ \ xA \rightarrow\ \ x\Box A$

Proof:

Proof:	1.	$xA \rightarrow A$	*QL axiom*
	2.	$\Box(\ \ xA \rightarrow A)$	*1, N*
	3.	$\Box(\ \ xA \rightarrow A) \rightarrow (\Box\ \ xA \rightarrow \Box A)$	*K*
	4.	$\Box\ \ xA \rightarrow \Box A$	*2,3, PL*
	5.	$x(\Box\ \ xA \rightarrow \Box A)$	*4, QL*
	6.	$x(\Box\ \ xA \rightarrow \Box A) \rightarrow\ \ x\Box\ \ xA \rightarrow\ \ x\Box A$	*QL axiom*
	7.	$x\Box\ \ xA \rightarrow\ \ x\Box A$	*5, 6, PL*
	8.	$\Box\ \ xA \rightarrow\ \ x\Box\ \ xA$	*QL axiom*
	9.	$\Box\ \ xA \rightarrow\ \ x\Box A$	*7, 8, PL*

This ensures that the domain of every possible world includes the domain of the given world. [Otherewise $\mathbf{BF_c}$ might be false, for even if everything in every world were *A*, there could be something in the actual world that fails to exist (hence to be *A*) at some world.]

— If we have equality, this half of the "fixed domain" idea is also captured by the following valid wff:

NecEx $\quad x\Box\ \ y(y = x)$

◆ REMARK 4

— Note that we can read **BF** as asserting that if a necessity holds *de re*, then it also holds *de dicto*. But we may as well read it as asserting that if a possibility holds *de dicto*, then it also holds *de re*—for **BF** is equivalent to:

BF◇ $\quad \Diamond\ \ xA \rightarrow\ \ x\Diamond A$

— In any case, **BF** and **BF**◇ are controversial. Consider the following instances:

- *If everything is necessarily spatio-temporally located, then necessarily everything is spatio-temporally located.*
- *If Berlusconi might have had a sister, then there is something that might have been Berlusconi's sister.*

— Likewise, we can read $\mathbf{BF_c}$ as asserting that if a necessity holds *de dicto*, then it holds *de re*. But we may as well read it as asserting that if a possibility holds *de re* it holds *de dicto*:

$\mathbf{BF_c}$◇ $\quad x\Diamond A \rightarrow \Diamond\ \ xA$

— $\mathbf{BF_c}$ and $\mathbf{BF_c}$◇ are more plausible. But they are also controversial. Consider:

- *If necessarily everything is self-identical, then everything is necessarily self-identical.*
- *If there is something that might not have existed, then it is possible for there to be something that doesn't exist.*

— In general, the problem with the "fixed domain" approach is that it seems a fundamental feature of common ideas about modality (at least: alethic modality) that the existence of many things is contingent, and that different objects exist in different possible worlds.

— However, one can avoid this objection by interpreting the quantifiers as ranging over all possible objects and letting a designate predicate 'E' express existence.

For all existing x: A	$x(\mathrm{E}x \rightarrow A)$
For some existing x: A	$x(\mathrm{E}x\quad A)$

BF and $\mathbf{BF_c}$ would then be non-problematic if interpreted unrestrictedly as holding of *possibilia*, and their existentially restricted versions would not be theorems and, therefore, would not give rise to the above objections

$x\Box A \rightarrow \Box\ \ xA$	$x(\mathrm{E}x \rightarrow \Box A) \rightarrow \Box\ \ x(\mathrm{E}x \rightarrow A)$
$\Box\ \ xA \rightarrow\ \ x\Box A$	$\Box\ \ x(\mathrm{E}x \rightarrow A) \rightarrow\ \ x(\mathrm{E}x \rightarrow \Box A)$

3. Semantics # 2: The world-relative approach

◆ MODELS

A model is a five-tuple $\mathcal{M} = \; W, R, D, V, Q \;$ such that:

— W, R, D, V are as in #1

— Q is a function assigning a non-empty set $Q(\alpha) \quad D$ to each world $\alpha \quad W$ (the domain of quantification for α), with the proviso $Q(\alpha) \quad Q(\beta)$ whenever $\alpha R \beta$.

(This proviso ensures that $\mathbf{BF_c}$ comes out valid, for recall: $\mathbf{BF_c}$ is derivable in quantified K).

Assignments are defined as in #1.

◆ TRUTH AND VALIDITY

The truth conditions of a wff A at a world α under an assignment μ are defined as follows:

1. $\models^{\mathcal{M}}_{\alpha} \mathbb{P}t_1...t_n$ [μ] iff $\mu^*(t_i) \quad Q(\alpha)$ for each i and $\quad \alpha, \mu^*(t_1), ..., \mu^*(t_n) \quad V(\mathbb{P})$

2–4 as in #1.

5. $\models^{\mathcal{M}}_{\alpha} \quad xA$ [μ] iff $\models^{\mathcal{M}}_{\alpha} A$ [μ'] for every x-alternative of μ' of μ such that $\mu'(x) \quad Q(\alpha)$

Other semantic notions (truth, validity in a model $\mathcal{M}$, validity in a class of models C, and logical validity) are defined as before.

◆ REMARKS

— The expressiveness problem persists.

— The validity of $\mathbf{BF_c}$, and more generally of the proviso that $Q(\alpha) \quad Q(\beta)$ whenever $\alpha R \beta$, is still controversial. Can we give it up?

— We can give it up, but the resulting semantics does not validate classical logic. We need supplement K with the axioms & rules of Free Logic. In particular, the QL axiom

$$xA \rightarrow A[t/x]$$

[which was used in the proof of $\mathbf{BF_c}$.] must be weakened to:

$$y(\quad xA \rightarrow A[y/x])$$

or, if we have equality, to:

$$(\quad xA \qquad y(y = t)) \rightarrow A[t/x]$$

— If we do so, then (as expected) $\mathbf{BF_c}$ is valid in a model iff the *inclusiveness* condition

$$\alpha R \beta \qquad Q(\alpha) \quad Q(\beta)$$

is explicitly imposed upon R.

— If we also require R to be *symmetric*, then **BF** is valid too.

4. Identity and Counterparthood

◆ Axioms

— If the language includes the identity predicate =, the standard identity axioms hold:

ID1	$t = t$
ID2	$t = t' \rightarrow (A \rightarrow A[t'/t])$

— As a consequence, we immediately get the following logical truths (provable regardless of whether we rely on a Classical or Free Logic for the quantifiers):

NecId	$t = t' \rightarrow \Box t = t'$
NecDiv	$t \neq t' \rightarrow \Box t \neq t'$

◆ Identity or counterparthood?

— Both types of semantics assume that objects may exist in more than one world, and that names are rigid designators.

— But one could object to this view on philosophical grounds. One could say that when we counterfactualize about someone, say John, we don't look at the ways John could have been but, rather, at the way John's *counterparts* are—his counterparts being those individuals that somehow correspond to John in other worlds.

This means that a statement such as 'John could have been a singer' would be true, not if there is a possible world in which John *is* a singer, but if there is a counterpart of John's, in some possible world, who *is* a singer.

— We can account for this intuition as follows (though a fuller account will be seen in part **3.2**, devoted to David Lewis's Counterpart theory):

◆ MODELS

A model is a six-tuple $\mathcal{M} = W, R, D, V, Q, C$ such that:

— W, R, D are as in #1 and #2
— Q is as in #2, but with the proviso that $Q(\alpha) \cap Q(\beta) =$ for all distinct α, β W.
— C is a binary relation (of counterpart) on D.
— V is a function assigning

- a function $V(a)\colon W \rightarrow D$ to every name a, with the proviso that $V(a)(\alpha)$ $Q(\alpha)$. (Intuitively, $V(a)(\alpha)$ is the individual denoted by 'a' in α.)
- a relation $V(\mathbb{P})$ $W \times D^n$ to every *n-ary* predicate $\mathbb{P}$. (If $\mathbb{P}$ is 0-ary, $V(\mathbb{P})$ is just the *proposition* P expressed by $\mathbb{P}$, as before; otherwise $V(\mathbb{P})$ is the *intension* of $\mathbb{P}$.)

Assignments are defined as in #1 and #2.

◆ TRUTH AND VALIDITY

The truth conditions of a wff A at a world α under an assignment μ are defined as follows:

1–3 as in #2.

4. $\models^{\mathscr{M}}_{\alpha} \Box A$ [μ] iff $\models^{\mathscr{M}}_{\beta}$ $y_1 \ldots$ $x_n(\mathrm{C}x_1t_1$ $\ldots$ $\mathrm{C}x_nt_n \rightarrow A[x_1/t_1 \ldots x_n/t_n])$ [μ] for every β W such that αRβ — where $t_1 \ldots t_n$ are all the names and variables occurring free in A.

5. as in #2.

Other semantic notions (truth, validity in a model $\mathscr{M}$, validity in a class of models C, and logical validity) are defined as before.